

Phase space embedding of electrocardiograms

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We study properties of the human electrocardiogram under the working hypothesis that fluctuations beyond the regular structure of single cardiac cycles are unpredictable. Against this background we discuss the possibility of using the phase space embedding method for this kind of signal. In particular, the specific nature of the stochastic or high-dimensional component allows us to use phase space embeddings for certain signal processing tasks. As practical applications, we discuss noise filtering, fetal ECG extraction, and the automatic detection of clinically relevant features. The main purpose of the paper is to connect results of embedding theory that have not been previously applied in practice, and practical applications that have not yet been justified theoretically. [S1063-651X(98)10911-X]

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I. INTRODUCTION

In nonlinear time series analysis, methods developed in nonlinear dynamics are applied to time series data in order to capture as much of the underlying structure as possible. One central question is whether the dynamics of the system can be reconstructed on the basis of the data given. Given the case in which the considered system is deterministic, the dynamics unambiguously evolves one state in phase space into another. Therefore, the reconstruction of phase space is a fundamental problem that plays the key role in many applications: as long as the state space reconstruction remains unjustified all of the consequent analysis may be wrong. From this point of view, we discuss the phase space embedding of realistic signals in the example of the normal human electrocardiogram. An introduction to nonlinear dynamics is found in [1,2]. An account of nonlinear time series methods is given by [3].

The electrocardiogram (ECG) is one of the most prominent clinical tools to monitor the activity of the heart. In order to record an electrocardiographical signal, metal electrodes are placed on the patient's chest wall and extremities (for details see, e.g., [4]). The potentials are generated by the atrial and ventricular muscle fibers. Due to the placement of the electrodes on the skin at some distance from the heart, the signals measured correspond to action potentials that are averaged over large regions of tissue. The spreading of the electrical activity over the cardiac muscle is controlled by the conduction system of the heart. Atrial and ventricular contraction and relaxation, respectively, correspond to characteristic ECG waves that are traditionally labeled in alphabetic order beginning with the letter *P*. As long as the average heart rate does not change dramatically, a nonpathological ECG shows a nearly periodic structure, which is due to the continuous generation of action potentials and the fixed pattern according to which the electrical activity spreads out over the cardiac muscle. However, apart from the deterministic structure some kind of variability can be found. On one hand, the length of the time interval between successive beats fluctuates—to a certain extent—around the mean heart rate. (In the medical literature the inter-beat-interval is often called the *RR* interval, which is defined as the time between

two consecutive *R* waves.) On the other hand, a slight variation of the cycle shape and amplitude can be observed. Both kinds of fluctuations can be gathered from Fig. 1, where two different cycles of the same ECG sequence are shown. As one can see in Fig. 2, part of the variation of the heart rate (middle trace) of a human at rest can be connected with the breath cycle (upper trace). (Data from the Santa Fe Institute time series contest [5].) The lower trace contains a randomized sequence that has the same autocorrelation structure, and the same cross correlation to the breath rate as the middle trace. For details regarding the generation of these surrogate data, and further references, see Ref. [6]. These random surrogate data explain most but not all of the underlying structure by linear correlations. One explanation for the remaining structure might be the presence of a high-dimensional or nonlinear stochastic component. In this paper we adopt as a working hypothesis that the fluctuations in the instantaneous heart rate are effectively unpredictable.

The Fourier domain is inappropriate to capture the structure described above, since the variation in cycle length leads to a dominating broad band component. An alternative would be to formulate the dynamics of the ECG cycle by a stochastically driven model in a low dimensional phase space. Since this space is not fully accessible by measurements, we propose the use of the delay reconstruction technique [7] as a convenient tool for revealing both the regular and stochastic aspects of the electrocardiogram. Using this technique, one attempts to reconstruct the state variables ξ of the system represented in phase space on the basis of the

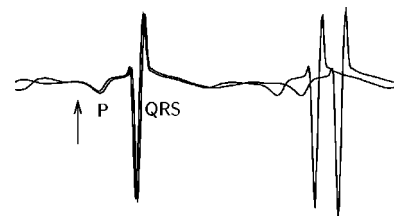


FIG. 1. Two different cycles of the same ECG have been aligned (indicated by the arrow) right before the beginning of the *P* waves. The stochastic components that lead to deviations from a pure limit cycle behavior manifest themselves in different cycle lengths and a slight variation of shape.

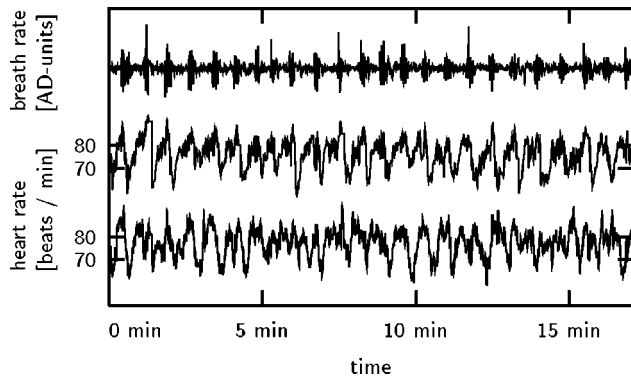


FIG. 2. Simultaneous measurements of breath and heart rates, upper and middle trace. The lower trace contains a randomized sequence preserving the autocorrelation structure and the cross correlation to the fixed breath rate series. The surrogate data mimic much of the structure contained in the data, but not all.

single lead measurement. Phase space reconstruction has been developed for purely deterministic systems. We do not want to make this assumption here. We will argue, however, as to why the technique can be appropriate in this case despite the stochastic component of the signal.

Delay reconstruction of electrocardiograms

In many cases of practical interest, such as the ECG, it is not possible to measure the state variables of a system directly. Instead, the measuring procedure yields some value $x = \varphi(\xi)$, when the system is in state ξ . Here, φ is a measurement function that in general, depends on the state variables in a nonlinear way. The time evolution of the state of the system results in a scalar time series x_1, x_2, x_3, \dots . In order to reconstruct the underlying dynamics in phase space, delay embedding techniques are commonly used. Delay vectors $\mathbf{x}_n = (x_n, x_{n-l}, x_{n-2l}, \dots, x_{n-(d-1)l})$ are a convenient method of transforming the scalar time series into d -dimensional vectors. Here, d corresponds to the embedding dimension while l is the lag between the time series elements. In the case of purely deterministic systems, the embedding theorem by Takens [7] and its generalization by Sauer, Yorke, and Casdagli [8] assure the equivalence between the reconstruction space and the original phase space of the system under fairly broad conditions. Figure 3 shows a two-dimensional reconstruction of the first 5000 time series

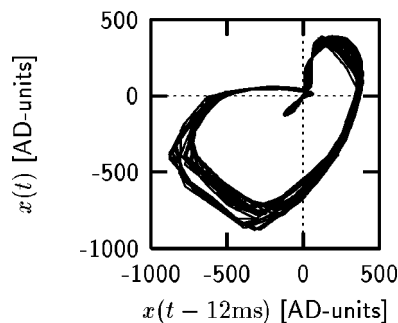


FIG. 3. Two dimensional delay reconstruction with delay time 12 ms using 5000 data points of the whole ECG sequence from which two cycles are shown in Fig. 1.

elements (20s) of the ECG sequence underlying Fig. 1 by using delay coordinates at a lag of 12 ms.

While for the signal processing applications discussed later usually much higher embedding dimensions are needed, even the two-dimensional representation illustrates the underlying phase space structure succinctly. It is remarkable that the signal structure in phase space is close to a manifold (which will be exploited for noise reduction purposes discussed later). This suggests that the underlying generating process is basically deterministic, which manifests itself in regular cycles, while the stochastic degrees of freedom lead to deviations from the basic cycle. While the fluctuations in cycle shape are indicated in the delay reconstruction, the stochastic variation of the time between cycles cannot be seen directly. This can be easily understood in the context of the following simplifying model. Let us suppose that the onset of a new cycle is started by an external trigger signal and that the RR -interval time fluctuates stochastically around the mean heart rate. Let us further assume that the ECG signal rests on the base line after a cycle is finished, until a new trigger signal starts the next depolarization. Since the constant base line corresponds to a single point in phase space, the stochastic variation of the cycle length cannot be revealed using delay vectors, provided they span a time shorter than the interbeat times. In order to reconstruct the underlying dynamics fully, it is necessary to use some additional information, such as the trigger signal, describing the onset of the next cycle.

To conclude this introduction, the delay reconstruction technique can help to reveal aspects of the nature of the underlying process, but it is not clear *a priori* that the dynamics of the system can be reconstructed by using delay coordinates in the case of stochastic forcing, even if the embedding dimension is chosen sufficiently large. Fortunately, the embedding theorems proven recently by Stark and co-workers [9,10] (in the following we refer to these theorems by the initials of the authors: SBDO) generalize the well known Takens embedding theorem [7] and provide results for the phase space reconstruction of stochastically forced dynamical systems. On the basis of the SBDO theorem it is possible to reconstruct the higher dimensional phase space from the scalar time series data by using delay coordinates when the sequence of stochastic influences that act on the system is known. Of course, in the case of the ECG, we don't know the driving process. However, we will argue that the necessary information can be recovered from the signal *a posteriori*.

II. EMBEDDING OF STOCHASTICALLY DRIVEN SIGNALS

In this section we give an overview over the fundamental notions introduced in the papers by Stark and co-workers [9,10] in order to state the key points of the essential embedding theorem for stochastic systems. We try to simplify the notation used as far as possible. The statement of the theorem as well as the formalism introduced are illustrated in the examples discussed subsequently.

In order to describe stochastic forcing, Stark considers skew product systems (to avoid confusion we denote the state variables of the system by ξ while \mathbf{x} labels the delay

vectors built from the scalar time series measurements in order to reconstruct the phase space of the system):

$$\begin{aligned}\xi_{n+1} &= \mathbf{f}(\xi_n, \Omega_n), \\ \Omega_{n+1} &= \sigma(\Omega_n).\end{aligned}$$

The dynamical system \mathbf{f} is stochastically driven by the bi-infinite sequence $\Omega = \dots \omega_{-2} \omega_{-1} \omega_0 \omega_1 \omega_2 \dots$, where Ω is built up by symbols ω_i that influence \mathbf{f} . The time evolution of the stochastic sequence Ω is realized by the map σ shifting the elements of Ω to the left by one position. In each time step, the dynamics \mathbf{f} is affected only by the central element ω_0 of Ω . It is natural to express the dependence of \mathbf{f} on ω_n by a sequence of maps $\mathbf{f}_n(\xi) = \mathbf{f}(\xi, \omega_n)$, so that the dynamical system becomes

$$\xi_{n+1} = \mathbf{f}_n(\xi_n). \quad (1)$$

In this framework it is possible to modify the underlying dynamics in each time step. This formalism will be illuminated in an example given below where the phase space reconstruction of a circular motion with fluctuating radius is discussed.

The central theorem of [10] addresses faces the question of whether it is possible to reconstruct the phase space of stochastically driven systems only on the basis of the time series. In analogy to the usual embedding techniques, one can define the d -dimensional delay embedding map $\Phi: M \rightarrow \mathbb{R}^d$ for skew product systems by

$$\Phi_\omega(\xi_n) = (\varphi(\xi_n), \varphi(\xi_{n+1}), \dots, \varphi(\xi_{n+d-1})) \quad (2)$$

(the use of the term ‘‘prelay’’ rather than ‘‘delay’’ vectors is for technical reasons). Here, the index ω has to be kept since the future of ξ_n depends explicitly on the stochastic sequence Ω under consideration. This can be stressed by writing

$$\begin{aligned}\Phi_\omega(\xi_n) &= (\varphi(\xi_n), \varphi(\mathbf{f}_n(\xi_n)), \dots, \varphi(\mathbf{f}_{n+d-2}(\dots \mathbf{f}_{n+1}(\mathbf{f}_n(\xi_n)) \dots))), \\ & \quad (3)\end{aligned}$$

where we have made use of Eq. (1). Essentially, the main theorem (No. 3.5 [10] and No. 7 [9], respectively) says that if $d \geq 2d_0 + 1$, then there is a *residual set* of dynamical systems \mathbf{f} and measurement functions φ , so that the delay embedding map Φ_ω yields an embedding for almost every stochastic sequence ω . For the detailed genericity conditions on \mathbf{f} , φ , and ω we refer to [9,10].

The crucial point here—compared with the embedding theorems by Takens [7] and Sauer, Yorke, and Casdagli [8]—is that it is *not* possible to reconstruct the underlying dynamics fully on the basis of only the measured time series, even if the embedding dimension d is sufficiently large and the dynamical system \mathbf{f} as well as the measurement function φ are generic. In order to take the stochastic nature of the process into account, it is necessary to use additional information describing the stochastic influences. To be precise, the knowledge of the underlying stochastic sequence is essential for the definition of the delay embedding map (3).

As an illustrative example consider the map

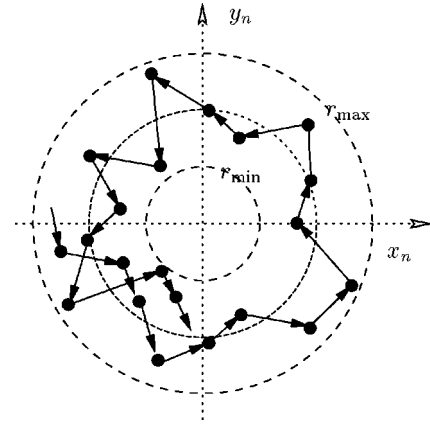


FIG. 4. Illustration of the phase space structure underlying equation (4). After many iterations, the dots evenly fill the disk area between r_{\min} and r_{\max} .

$$\begin{aligned}\underbrace{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}}_{\xi_{n+1}} &= \underbrace{\frac{r_{n+1}}{r_n} \mathbf{R}_{[\phi]}}_{\mathbf{f}_n} \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix}}_{\xi_n} \\ &= r_{n+1} \mathbf{R}_{[(n+1)\phi]} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\end{aligned} \quad (4)$$

where \mathbf{R}_ϕ is a rotation by ϕ ,

$$\mathbf{R}_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix},$$

and the radius r_n is chosen at random between r_{\min} and r_{\max} while the rotation angle ϕ is fixed and generically incommensurate with 2π .

For simplicity, let the measurement function φ ‘‘measure’’ the first coordinate:

$$\varphi(\xi_n) = x_n.$$

The central question is now whether the scalar time series $x_0, \dots, x_n, x_{n+1}, \dots$ yields an appropriate basis to reconstruct the dynamics of the system. Figure 4 contains a schematic representation of the phase space structure underlying equation (4). Since the fluctuations of the radius are bounded by r_{\min} and r_{\max} , the points generated according to Eq. (4) are sprinkled over a disk. This disk corresponds to the compact manifold M in the theorems by Stark and co-workers.

For a moment consider the case that the sequence of radii would be fixed to one: $r_n \equiv 1$. In this specific case it is obvious that the two-dimensional delay reconstruction map $\Phi(\mathbf{x}_{n-1}) = (x_{n-1}, x_n)$ —which is based on the measurement function specified above—yields an embedding for almost every value of ϕ . This is simply due to the fact that a uniform motion on a circle can easily be reconstructed by its first Cartesian coordinate since both coordinates are related by a constant phase shift. As far as a *deterministic* variation

of the radius is involved—according to Takens’s theorem—at least three-dimensional delay embedding maps are needed to avoid self-intersections in reconstruction space. It can be seen quickly that a pure delay embedding technique in which the dimension takes into account only the degrees of freedom of \mathbf{f} is, in general, not sufficient to reconstruct the underlying dynamics in the case of *stochastic forcing*. In contrast to deterministic systems, due to the stochastic influences crossing points can appear between the trajectories on the manifold M as indicated in Fig. 4. In order to determine the continuation of the trajectories, it is necessary to use some information beyond the delay coordinates. Fortunately, as Stark points out in the theorem cited above, the additional knowledge of the driving stochastic sequence allows the full reconstruction of the underlying dynamics. Referring to map (4), the knowledge of the sequence of radii r_n allows the continuation of the trajectories beyond the crossing points.

Apart from the pure illustrating function regarding the basic idea underlying Stark’s theorem, the model discussed can help to motivate the embedding of more realistic signals such as ECG sequences. As indicated in Fig. 3, the delay reconstruction of a normal human electrocardiogram bears resemblance to a limit cycle structure with fluctuations. As pointed out in the Introduction, the fluctuations affect both the cycle length and its shape. One natural way to describe the stochastic influences would be by trying to observe the driving input signal. However, the variable parameters that influence the ECG wave form are not usually accessible to measurements and in fact, are not fully known. This is a general problem with the approach behind the work of Stark and co-workers, as well as that of Casdagli [11]: regarding a system as an input-output device is only useful for the analysis of time series if both the output and the input sequences are observed.

One of the main ideas we want to put forth in this paper is that there are situations in which the input sequence can to some extent be inferred from the observed data. In such a case, the embedding procedure may be applied successfully, on the theoretical foundation of the theorem discussed above. In the case of the ECG, the stochastic driving mainly affects the duration of the cycles, which is accessible a posteriori by measuring the *RR* intervals. If there is a variation of the ECG wave form itself that can be captured by a few parameters, these can also be measured once the ECG has been recorded. The fact we use in doing this is that ECG recordings contain considerable redundancy. With regard to ECG signals this implies that the explicit specification of the underlying sequence Ω is not strictly necessary to make use of the embedding technique.

Let us study one more toy model to illustrate how the lacking information about the input sequence may be contained in a signal from an input-output system. Consider a damped harmonic oscillator,

$$\ddot{x} + \dot{x} + x = a(t), \tag{5}$$

where the driving term is zero except for kicks of random strength at times t_i such that the interbeat intervals, $t_i - t_{i-1}$ are random in the interval $[p, q]$. Figures 5 and 6 show two trajectories of such a system with different choices of the interbeat time interval $[p, q]$. On discretization, the

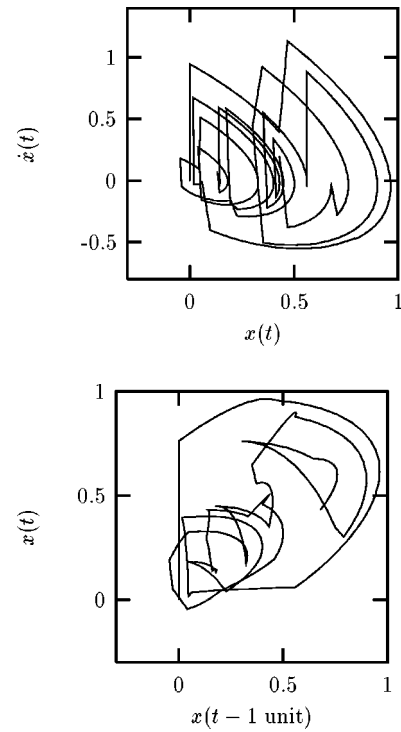


FIG. 5. Trajectories of a kicked, damped harmonic oscillator. Upper: true phase space. Lower: delay embedding. Kicks occur close in time.

kicks are realized by finite jumps by a random amount in the interval $[0, 1]$. The upper panel in each of these figures shows the true phase space, while in the lower panels a delay representation is used with a delay of one time unit. In Fig. 5,

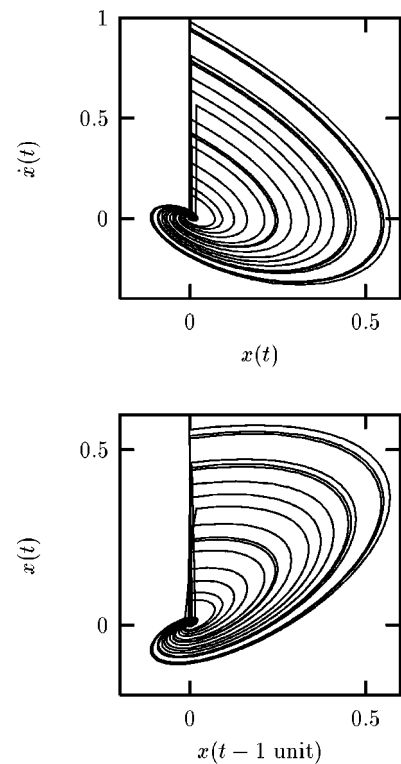


FIG. 6. Same as Fig. 5, but kicks are well separated in time such that the system can relax between kicks.

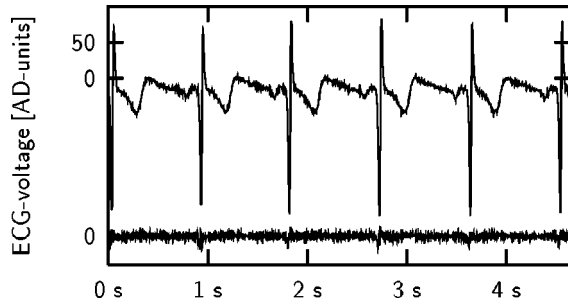


FIG. 7. Deviations between the ECG sequence [12] (upper trace) and one-step predictions generated according to Eq. (6). The signals are measured in uncalibrated AD units.

beats are initiated with a time separation of $p=0$ to $q=T/2$ where $T=4\pi/\sqrt{3}\approx 7.26$ time units is the period of oscillation. This does not allow the system to relax sufficiently between kicks in order to form characteristic structure in phase space. Consequently, an embedding provides no clear picture. In Fig. 6, no kicks were closer in time than $p=T$, the maximal separation being $q=3T$. The interbeat parts of the trajectory are distinct because of the different kick strength, but since this is the only fluctuating parameter, they are essentially restricted to a two-dimensional manifold. This manifold is preserved under time delay embedding although neither the sequence of beat times nor the beat amplitudes are used explicitly. The randomness prevails in the indeterminacy at the origin as to when the next beat will occur.

III. PREDICTING AND PROJECTING ECG SIGNALS

While the structure of a normal human ECG is well predictable on a scale considerably shorter than one cycle, fluctuations in cycle length limit the accuracy at the onset of the QRS complexes. Therefore, large deviations can be found between the original ECG data and the one-step predictions generated as soon as a new QRS complex appears, while the difference between both signals is usually comparable to the magnitude of the noise level. Figure 7 illustrates these deviations with the ECG sequence plotted. The ansatz chosen makes explicit use of the phase space reconstruction by delay vectors \mathbf{x}_n . The ECG data considered are highly over-sampled so that x_{n+1} is close to x_n up to some corrections that are modeled by radial basis functions:

$$x_{n+1} = x_n + \underbrace{\sum_{k=1}^K a_k \exp(-b[\mathbf{x}_n - \mathbf{c}_k]^2)}_{F(\mathbf{x}_n)} + C. \quad (6)$$

The constant C handles a zero offset directly instead by fitting via radial basis functions, which would affect the quality of the parameters a_k , b , and c_k . In order to place the centers c_k of the basis functions, a grid-based algorithm was used. The result plotted in Fig. 7 was achieved by aspiring a uniform density of centers over the phase space regions occupied by the reconstruction vectors and a total number of 10 basis functions. Increasing the number of basis functions

leads to a better approximation of the ECG signal and therefore a lower in-sample prediction error. In Fig. 7 the prediction error during the interbeat times is comparable to the magnitude of the original ECG data noise level. It is remarkable that the error of the one-step predictions is much larger at the onset of the QRS complexes so that the overall distribution is nonuniform. Therefore one might conclude that the onset of the next QRS complex remains unpredictable.

To reduce the uncertainty underlying the onset of the next QRS complex one could think of using the length of the following RR interval as an additional input signal. It might be possible to lower the large prediction error at the beginning R wave by this technique, since the moment of the spikelike deflection is now completely determined. The use of an additional input sequence that characterizes the stochastic degrees of freedom corresponds to the fixing of the stochastic sequence Ω in the formalism introduced by Stark. Since the prediction of ECG signals is not in itself of much practical relevance, we do not describe further details. The concrete implementation is technically involved and would go far beyond the scope of this paper.

State space reconstruction techniques form an essential tool for noise reduction purposes. In order to separate the signal from its noise components, basically two different techniques can be pursued. (I) On one hand, a global fit to the dynamical equations can be used for the reconstruction of a noiseless trajectory of the system, as has been done in [13], for example. In this context, state space reconstruction techniques are involved to capture the underlying dynamics. For example in Eq. (6) the reconstruction technique enters via the delay vectors \mathbf{x}_n . As an iterative method for reaching a trajectory that obeys the reconstructed dynamics, gradient descent methods have been investigated in [13]. (II) On the other hand, it is possible to exploit the local structure in the reconstruction space directly for noise reduction purposes without referring to the global underlying dynamics. Methods based on this local projective approach have been proposed in [14]; see [15] for a review. The idea is that if it is found empirically that the data points in reconstruction space are located close to a manifold, the error by projecting onto that manifold may be smaller than the error due to the noise. If the trajectory of the system lies on a low-dimensional attractor, the projection technique can indeed be shown to reduce noise. But also for nondeterministic signals, nonlinear noise reduction techniques have been successfully applied for signal processing tasks such as ECG noise reduction [16] or fetal ECG extraction [17,18]. The basic reason why this works is the mechanism illustrated in Fig. 6. These signals, in spite of being nondeterministic, nevertheless are near a low-dimensional manifold. Let us give the extraction of the fetal ECG as an application of the delay embedding of ECG signals (fetal electrocardiography is the only method to monitor the cardiac activity of the fetus in a noninvasive way). Figure 8 contains a two-dimensional delay representation of an abdominal ECG recording (left panel) and the reconstructed maternal manifold structure (right panel) by the projective noise reduction approach. Segments of the corresponding time series are shown in Fig. 9 (data by courtesy of Hofmeister [19]). The upper trace contains the abdominal signal (electrodes have been placed on the maternal abdomen to record most of the fetal heart activity) while the middle

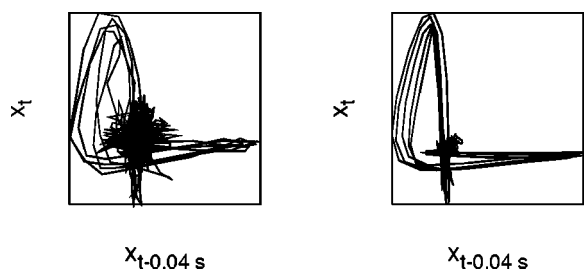


FIG. 8. Delay reconstruction of the abdominal ECG recording (left panel) and the maternal component (right panel).

trace shows the abdominal projection of the maternal electrocardiogram that has been extracted by noise reduction. The difference between the upper and middle trace yields the noisy fetal component, which can be cleaned up in a second noise reduction step (lower trace). The extraction of the fetal electrocardiogram is broadly described in [20], for a powerful modification of the projective algorithm that allows the real time extraction of the fetal electrocardiogram on a laptop PC (Pentium processor at 133 MHz, Linux operating system, 250 Hz sampling rate) see [21].

As another practical application, phase space reconstruction techniques can be employed for the localization of fundamental ECG waves such as P , R , or T waves. Instead of comparing the structures directly on the basis of the time series, similar ECG waves can be identified as adjacent curves in reconstruction space. For example, the smaller loop to the lower left side of the origin in Fig. 3 corresponds to the P waves of the full ECG sequence partially plotted in Fig. 1, while the larger framing loop is due to the QRS complex. For the automatic detection of ECG cycle substructures, delay vectors are defined to be neighbors if their distance to the vector under consideration is smaller than a given size (for neighbor searching methods cf., e.g., [3]). Nearby trajectories can be identified in phase space by searching for neighbors of the reconstruction vectors of a specified structure. On the basis of the original time series, these trajectories correspond to segments that are similar to the specified one. By tagging a P , R , or T wave of a given electrocardiogram, previous or following waves of this type can be located automatically.

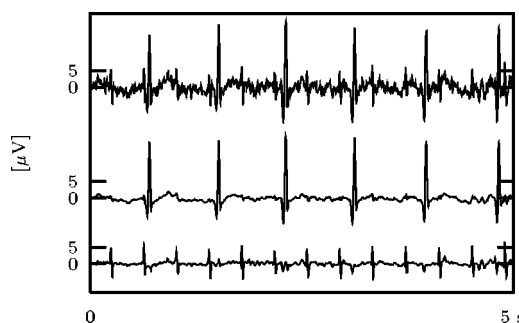


FIG. 9. Fetal ECG extraction by locally linear projections in phase space. The abdominal recording (upper trace) contains a dominant maternal component and the fetal signal, which is covered by noise. The first noise reduction step yields the reconstructed maternal signal (middle), which allows to separate the noisy fetal ECG from the original recording. A second projection can be applied to clean up the fetal component (lower trace).

To conclude, embedding techniques form a fundamental starting point for many applications. Therefore, it is desirable to motivate the reconstructability of the phase space in the case of typical time series. We illuminated this issue by considering the example of the human electrocardiogram. Under the working hypothesis that fluctuations beyond the regular structure of the cardiac cycle are unpredictable we found the embedding theorems of Stark and coworkers [9,10] to be useful to motivate the embedding of ECG data. Finally, we discussed useful applications that are based on phase space reconstructions of the electrocardiogram in order to give examples of how embedding techniques can be employed for practical tasks.

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- [1] D. Kaplan and L. Glass, *Understanding Nonlinear Dynamics* (Springer-Verlag, New York, 1995).
 - [2] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).
 - [3] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis* (Cambridge University Press, Cambridge, 1997).
 - [4] A. L. Goldberger and E. Goldberger, *Clinical Electrocardiography* (Mosby, St. Louis, 1977).
 - [5] D. R. Rigney, A. L. Goldberger, W. Ocasio, Y. Ichimaru, G. B. Moody, and R. Mark, in *Time Series Prediction: Forecasting the Future and Understanding the Past*, edited by A. S. Weigend and N. A. Gershenfeld, Santa Fe Institute Studies in the Science of Complexity, Proceedings Vol. XV (Addison-Wesley, New York, 1993).
 - [6] T. Schreiber, *Phys. Rev. Lett.* **80**, 2105 (1998).
 - [7] F. Takens, *Detecting Strange Attractors in Turbulence*, Lecture Notes in Mathematics Vol. 898 (Springer-Verlag, New York, 1981), pp. 366–81.
 - [8] T. Sauer, J. Yorke, and M. Casdagli, *J. Stat. Phys.* **65**, 579 (1991).
 - [9] J. Stark, D. S. Broomhead, M. E. Davies, and J. Huke, *Nonlinear Analysis* **30**, 5303 (1997).
 - [10] J. Stark, *J. Nonlinear Sci.* (to be published).
 - [11] M. Casdagli, in *Nonlinear Modeling and Forecasting*, edited by M. Casdagli and S. Eubank, SFI Studies in the Sciences of Complexity (Addison-Wesley, Reading, MA, 1992).
 - [12] MIT-Beth-Israel Hospital Polysomnographic Database, Biomedical Engineering Centre (MIT, Cambridge, 1991).
 - [13] M. E. Davies, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **3**, 113 (1992).

- [14] P. Grassberger, R. Hegger, H. Kantz, C. Schaffrath, and T. Schreiber, *Chaos* **3**, 127 (1993).
- [15] E. J. Kostelich and T. Schreiber, *Phys. Rev. E* **48**, 1752 (1993).
- [16] T. Schreiber and D. T. Kaplan, *Chaos* **6**, 87 (1996).
- [17] T. Schreiber and D. T. Kaplan, *Phys. Rev. E* **53**, R4326 (1996).
- [18] M. Richter, thesis, University of Wuppertal, 1997 (in German).
- [19] J. F. Hofmeister, J. C. Slocumb, L. M. Kottmann, J. B. Picchiottino, and D. G. Ellis, *Biomed. Instr. Technol.*, p. 391 (Sept. 1994).
- [20] M. Richter, T. Schreiber, and D. T. Kaplan, *IEEE Trans. Biomed. Eng.* **45**, 133 (1998).
- [21] T. Schreiber and M. Richter, Report No. WUB 98-8.